

Some considerations in relation to the matrix equation $AXB = C$

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Abstract

In this paper we represent a new form of condition for the consistency of the matrix equation $AXB = C$. If the matrix equation $AXB = C$ is consistent, we determine a form of general solution which contains both reproductive and non-reproductive solutions. Also, we consider applications of the concept of reproductivity for obtaining general solutions of some matrix systems.

Keywords:

Matrix equation $AXB = C$, reproductive equation, reproductive solution

1. Introduction

The concept of the reproductive equations was introduced by S.B. Prešić [4] in 1968. In this part of the paper we give the definition of reproductive equations and the most important statements related to the reproductive equations. Using the concept of reproductivity in the next section we obtain the general solutions of some matrix systems.

Let S be a given non-empty set and J be a given unary relation of S . Then an equation $J(x)$ is *consistent* if there is at least one element $x_0 \in S$, so-called *the solution*, such that $J(x_0)$ is true. A formula $x = \phi(t)$, where $\phi : S \rightarrow S$ is a given function, represents *the general solution* of the equation $J(x)$ if and only if

$$(\forall t)J(\phi(t)) \wedge (\forall x)(J(x) \implies (\exists t)x = \phi(t)).$$

In this part of the paper we give the definition of reproductive equations and the fundamental statements related to the reproductive equations.

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Definition 1.1. *The reproductive equations are the equations of the following form:*

$$x = \varphi(x),$$

where x is a unknown, S is a given set and $\varphi : S \longrightarrow S$ is a given function which satisfies the following condition:

$$\varphi \circ \varphi = \varphi. \quad (1)$$

The condition (1) is called *the condition of reproductivity*. The fundamental properties of the reproductive equations are given by the following two statements S.B. Prešić [4] (see also [5], [6] and [18]).

Theorem 1.1. *For any consistent equation $J(x)$ there is an equation of the form $x = \varphi(x)$, which is equivalent to $J(x)$ being in the same time reproductive as well. ♦*

Theorem 1.2. *If a certain equation $J(x)$ is equivalent to the reproductive one $x = \varphi(x)$, the general solution is given by the formula $x = \varphi(y)$, for any value $y \in S$. ♦*

Let us remark that a formula $x = \phi(t)$, where $\phi : S \rightarrow S$ is a given function, represents *the reproductive general solution* [29] of the equation $J(x)$ if and only if

$$(\forall t)J(\phi(t)) \wedge (\forall t)(J(t) \implies t = \phi(t)).$$

S.B. Prešić was the first one who considered implementations of reproductivity on some matrix equations [4] (see also [2], [9]). The concept of reproductivity allows us to analyse various forms of the solution. General applications of the concept of reproductivity were also considered by J.D. Kečkić in [7], [8] and J.D. Kečkić and S.B. Prešić in [16].

2. The matrix equation $AXB = C$

Let $m, n \in \mathbb{N}$ and \mathbb{C} is the field of complex numbers. The set of all matrices of order $m \times n$ over \mathbb{C} is denoted by $\mathbb{C}^{m \times n}$. We use denotement $\mathbb{C}_a^{m \times n}$ for the set of all $m \times n$ complex matrices whose rank is a . Let $A = [a_{i,j}] \in \mathbb{C}^{m \times n}$. By A^T , $A_{i \rightarrow}$ and $A_{\downarrow j}$ we denote the transpose of A , the i -th row of A and the j -th column of A , respectively.

Therefore,

$$A_{i \rightarrow} = (a_{i,1}, a_{i,2}, \dots, a_{i,n}), i = 1, \dots, m$$

and

$$A_{\downarrow j} = (a_{1,j}, a_{2,j}, \dots, a_{m,j})^T, j = 1, \dots, n.$$

A solution of the matrix equation $AXA = A$ is called $\{1\}$ -inverse of A and it is denoted by $A^{(1)}$.

The matrix equation

$$AXB = C \quad (2)$$

was considered by many authors (see [13], [14], [17], [23]-[25], [33]). There are many papers (see [11], [12], [20], [22], [27]-[30], [32], [34]) where the matrix equation (2) is studied as a part of different matrix systems.

This part of the paper is organized as follows: In 2.1. we represent the new form of condition for the consistency of the matrix equation (2). An extension of Penrose's theorem related to the general solution of the matrix equation (2) is given in 2.2. Namely, we represent the formula of general solution of the matrix equation (2) if any particular solution X_0 is known. In 2.3. we give a new form of particular solution X_0 , using the result in 2.1., such that the formula of general solution of the matrix equation (2), which is given in 2.2., is reproductive. In 2.4. we give two applications on some matrix systems which are in relation to the matrix equation (2).

2.1. Let $A \in \mathbb{C}_a^{m \times n}$, $B \in \mathbb{C}_b^{p \times q}$ and $C \in \mathbb{C}^{m \times q}$. The matrix $A \in \mathbb{C}_a^{m \times n}$ has a linearly independent rows. If the linearly independent rows of the matrix A are not at the first a positions, by multiplying the matrix A by a permutation matrix T_A on the left, we can permute the rows of the matrix A .

Therefore, the matrix

$$\widehat{A} = T_A A \quad (3)$$

has linearly independent rows *at the first a positions*. Analogue, the matrix

$$\widehat{B} = B T_B \quad (4)$$

has linearly independent columns *at the first b positions*.

The considerations which follow are valid for any choice of matrices T_A and T_B such that \widehat{A} has linearly independent rows at the first a positions and \widehat{B} has linearly independent columns at the first b positions.

Let

$$\widehat{C} = T_A C T_B. \quad (5)$$

Next, let for the matrices A and B regular matrices Q_1, P_1, Q_2 and P_2 be determined such that the following equalities are true:

$$Q_1 A P_1 = E_a = \left[\begin{array}{c|c} I_a & 0 \\ \hline 0 & 0 \end{array} \right] \quad \text{and} \quad Q_2 B P_2 = E_b = \left[\begin{array}{c|c} I_b & 0 \\ \hline 0 & 0 \end{array} \right] \quad (6)$$

i.e.

$$A = Q_1^{-1} E_a P_1^{-1} \quad \text{and} \quad B = Q_2^{-1} E_b P_2^{-1}. \quad (7)$$

Then, from (3), (4) and (7) we get that

$$\widehat{A} = T_A Q_1^{-1} E_a P_1^{-1} \quad \text{and} \quad \widehat{B} = Q_2^{-1} E_b P_2^{-1} T_B$$

i.e.

$$\widehat{A} = (Q_1 T_A^{-1})^{-1} E_a P_1^{-1} \quad \text{and} \quad \widehat{B} = Q_2^{-1} E_b (T_B^{-1} P_2)^{-1}.$$

If we introduce the following designations:

$$\widehat{Q}_1 = Q_1 T_A^{-1} \quad \text{and} \quad \widehat{P}_2 = T_B^{-1} P_2 \quad (8)$$

we get that

$$\widehat{A} = \widehat{Q}_1^{-1} E_a P_1^{-1} \quad \text{and} \quad \widehat{B} = Q_2^{-1} E_b \widehat{P}_2^{-1}. \quad (9)$$

Considering that the following equalities are true for $A^{(1)}$ and $B^{(1)}$:

$$A^{(1)} = P_1 \left[\frac{I_a}{X_2} \middle| \frac{X_1}{X_3} \right] Q_1 \quad \text{and} \quad B^{(1)} = P_2 \left[\frac{I_b}{Y_2} \middle| \frac{Y_1}{Y_3} \right] Q_2 \quad (10)$$

where X_1, X_2, X_3, Y_1, Y_2 and Y_3 are arbitrary matrices corresponding dimensions (see C. Rohde [3]), then:

$$\widehat{A}^{(1)} = P_1 \left[\frac{I_a}{X_2} \middle| \frac{X_1}{X_3} \right] \widehat{Q}_1 \quad \text{and} \quad \widehat{B}^{(1)} = \widehat{P}_2 \left[\frac{I_b}{Y_2} \middle| \frac{Y_1}{Y_3} \right] Q_2$$

i.e.

$$\widehat{A} \widehat{A}^{(1)} = \widehat{Q}_1^{-1} E_a P_1^{-1} P_1 \left[\frac{I_a}{X_2} \middle| \frac{X_1}{X_3} \right] \widehat{Q}_1 = \widehat{Q}_1^{-1} \left[\frac{I_a}{0} \middle| \frac{X_1}{0} \right] \widehat{Q}_1 \quad (11)$$

and

$$\widehat{B}^{(1)} \widehat{B} = \widehat{P}_2 \left[\frac{I_b}{Y_2} \middle| \frac{Y_1}{Y_3} \right] Q_2 Q_2^{-1} E_b \widehat{P}_2^{-1} = \widehat{P}_2 \left[\frac{I_b}{Y_2} \middle| \frac{0}{0} \right] \widehat{P}_2^{-1}. \quad (12)$$

As we mentioned, the matrix \widehat{A} has linearly independent rows at the first a positions and the matrix \widehat{B} has linearly independent columns at the first b positions.

Let

$$\widehat{A}_{i \rightarrow} = \sum_{l=1}^a \alpha_{i,l} \widehat{A}_{l \rightarrow}, \quad i = a+1, \dots, m \quad (13)$$

and

$$\widehat{B}_{\downarrow j} = \sum_{k=1}^b \beta_{k,j} \widehat{B}_{\downarrow k}, \quad j = b+1, \dots, q. \quad (14)$$

for some scalars $\alpha_{i,l}$ and $\beta_{k,j}$. Then, \widehat{Q}_1 and \widehat{P}_2 have the following form:

$$\widehat{Q}_1 = \left[\frac{I_a}{L_1} \middle| \frac{0}{I_{m-a}} \right] \quad \text{and} \quad \widehat{P}_2 = \left[\frac{I_b}{0} \middle| \frac{L_2}{I_{q-b}} \right] \quad (15)$$

where

$$L_1 = \begin{bmatrix} -\alpha_{a+1,1} & \dots & -\alpha_{a+1,a} \\ \vdots & \dots & \vdots \\ \vdots & \dots & \vdots \\ \vdots & \dots & \vdots \\ -\alpha_{m,1} & \dots & -\alpha_{m,a} \end{bmatrix} \quad \text{and} \quad L_2 = \begin{bmatrix} -\beta_{1,b+1} & \dots & -\beta_{1,q} \\ \vdots & \dots & \vdots \\ \vdots & \dots & \vdots \\ \vdots & \dots & \vdots \\ -\beta_{b,b+1} & \dots & -\beta_{b,q} \end{bmatrix}.$$

Let us emphasize that the following statement is true.

Lemma 2.1. *Let $A \in \mathbb{C}_a^{m \times n}$, $B \in \mathbb{C}_b^{p \times q}$, $C \in \mathbb{C}^{m \times q}$. Suppose that \hat{A} and \hat{B} are determined by (3) and (4). Then, the conditions*

$$AA^{(1)}CB^{(1)}B = C \quad (16)$$

and

$$\hat{A}\hat{A}^{(1)}\hat{C}\hat{B}^{(1)}\hat{B} = \hat{C} \quad (17)$$

are equivalent.

Proof. $AA^{(1)}CB^{(1)}B = C \iff T_A^{-1}\hat{A}\hat{A}^{(1)}T_ACT_B\hat{B}^{(1)}\hat{B}T_B^{-1} = C$
 $\iff \hat{A}\hat{A}^{(1)}T_ACT_B\hat{B}^{(1)}\hat{B} = T_ACT_B \iff \hat{A}\hat{A}^{(1)}\hat{C}\hat{B}^{(1)}\hat{B} = \hat{C}. \blacklozenge$

In the following statement we give the necessary and sufficient condition so that the condition (16) is true for any choice of $\{1\}$ -inverses $A^{(1)}$ and $B^{(1)}$.

Theorem 2.1. *Let $A \in \mathbb{C}_a^{m \times n}$, $B \in \mathbb{C}_b^{p \times q}$, $C \in \mathbb{C}^{m \times q}$. Suppose that \hat{A} and \hat{B} are determined by (3) and (4) and that (13) and (14) are satisfied. Then, the condition (16) is true for any choice of $\{1\}$ -inverses $A^{(1)}$ and $B^{(1)}$ iff*

$$\hat{C} = \begin{bmatrix} c_{1,1} & \dots & c_{1,b} & \sum_{k=1}^b \beta_{k,b+1}c_{1,k} & \dots & \sum_{k=1}^b \beta_{k,q}c_{1,k} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ c_{a,1} & \dots & c_{a,b} & \sum_{k=1}^b \beta_{k,b+1}c_{a,k} & \dots & \sum_{k=1}^b \beta_{k,q}c_{a,k} \\ \sum_{l=1}^a \alpha_{a+1,l}c_{l,1} & \dots & \sum_{l=1}^a \alpha_{a+1,l}c_{l,b} & \sum_{l=1}^a \sum_{k=1}^b \alpha_{a+1,l}\beta_{k,b+1}c_{l,k} & \dots & \sum_{l=1}^a \sum_{k=1}^b \alpha_{a+1,l}\beta_{k,q}c_{l,k} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ \sum_{l=1}^a \alpha_{m,l}c_{l,1} & \dots & \sum_{k=1}^b \alpha_{m,l}c_{l,b} & \sum_{l=1}^a \sum_{k=1}^b \alpha_{m,l}\beta_{k,b+1}c_{l,k} & \dots & \sum_{l=1}^a \sum_{k=1}^b \alpha_{m,l}\beta_{k,q}c_{l,k} \end{bmatrix},$$

where $c_{i,j}$ are arbitrary elements of \mathbb{C} .

Proof. (\implies): Suppose that the condition (16) is valid for any choice of $\{1\}$ -inverses $A^{(1)}$ and $B^{(1)}$. Based on Lemma 2.1. the condition (17) is also valid. Then, considering the equalities (11) and (12), we get the following equality

$$\widehat{Q}_1^{-1} \left[\begin{array}{c|c} I_a & X_1 \\ \hline 0 & 0 \end{array} \right] \widehat{Q}_1 \widehat{C} \widehat{P}_2 \left[\begin{array}{c|c} I_b & 0 \\ \hline Y_2 & 0 \end{array} \right] \widehat{P}_2^{-1} = \widehat{C}.$$

By multiplying the previous equality by \widehat{Q}_1 on the left and by \widehat{P}_2 on the right we get

$$\left[\begin{array}{c|c} I_a & X_1 \\ \hline 0 & 0 \end{array} \right] \widehat{Q}_1 \widehat{C} \widehat{P}_2 \left[\begin{array}{c|c} I_b & 0 \\ \hline Y_2 & 0 \end{array} \right] = \widehat{Q}_1 \widehat{C} \widehat{P}_2. \quad (18)$$

Suppose that

$$\widehat{C} = \begin{bmatrix} c_{1,1} & \dots & c_{1,q} \\ \vdots & \dots & \vdots \\ \vdots & \dots & \vdots \\ c_{m,1} & \dots & c_{m,q} \end{bmatrix}.$$

We are going to show that \widehat{C} has the following form:

$$\widehat{C} = \begin{bmatrix} c_{1,1} & \dots & c_{1,b} & \sum_{k=1}^b \beta_{k,b+1} c_{1,k} & \dots & \sum_{k=1}^b \beta_{k,q} c_{1,k} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ c_{a,1} & \dots & c_{a,b} & \sum_{k=1}^b \beta_{k,b+1} c_{a,k} & \dots & \sum_{k=1}^b \beta_{k,q} c_{a,k} \\ \sum_{l=1}^a \alpha_{a+1,l} c_{l,1} & \dots & \sum_{l=1}^a \alpha_{a+1,l} c_{l,b} & \sum_{l=1}^a \sum_{k=1}^b \alpha_{a+1,l} \beta_{k,b+1} c_{l,k} & \dots & \sum_{l=1}^a \sum_{k=1}^b \alpha_{a+1,l} \beta_{k,q} c_{l,k} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ \sum_{l=1}^a \alpha_{m,l} c_{l,1} & \dots & \sum_{k=1}^b \alpha_{m,l} c_{l,b} & \sum_{l=1}^a \sum_{k=1}^b \alpha_{m,l} \beta_{k,b+1} c_{l,k} & \dots & \sum_{l=1}^a \sum_{k=1}^b \alpha_{m,l} \beta_{k,q} c_{l,k} \end{bmatrix}.$$

Let

$$E = \widehat{Q}_1 \widehat{C} \widehat{P}_2 \quad \text{and} \quad F = \left[\begin{array}{c|c} I_a & X_1 \\ \hline 0 & 0 \end{array} \right] E \left[\begin{array}{c|c} I_b & 0 \\ \hline Y_2 & 0 \end{array} \right]. \quad (19)$$

From (15) we obtain that

$$\begin{aligned} \text{for } i = 1, \dots, a, j = 1, \dots, b & \quad E_{i,j} = c_{i,j}, \\ \text{for } i = 1, \dots, a, j = b+1, \dots, q & \quad E_{i,j} = c_{i,j} - \sum_{k=1}^b \beta_{k,j} c_{i,k}, \\ \text{for } i = a+1, \dots, m, j = 1, \dots, b & \quad E_{i,j} = c_{i,j} - \sum_{l=1}^a \alpha_{i,l} c_{l,j}, \\ \text{for } i = a+1, \dots, m, j = b+1, \dots, q & \quad E_{i,j} = c_{i,j} - \sum_{l=1}^a \alpha_{i,l} c_{l,j} \\ & \quad - \sum_{k=1}^b \beta_{k,j} (c_{i,k} - \sum_{l=1}^a \alpha_{i,l} c_{l,k}) \end{aligned}$$

and

$$\begin{aligned} \text{for } i = 1, \dots, a, j = 1, \dots, b & \quad F_{i,j} = c_{i,j} + \sum_{\bar{l}=a+1}^m x_{i,\bar{l}} (c_{\bar{l},j} - \sum_{l=1}^a \alpha_{\bar{l},l} c_{l,j}) \\ & \quad + \sum_{\bar{k}=b+1}^q y_{\bar{k},j} [c_{i,\bar{k}} - \sum_{k=1}^b \beta_{k,\bar{k}} c_{i,k} \\ & \quad + \sum_{\bar{l}=a+1}^m x_{i,\bar{l}} \{ c_{\bar{l},\bar{k}} - \sum_{l=1}^a \alpha_{\bar{l},l} c_{l,\bar{k}} \\ & \quad - \sum_{k=1}^b \beta_{k,\bar{k}} (c_{\bar{l},k} - \sum_{l=1}^a \alpha_{\bar{l},l} c_{l,k}) \}], \end{aligned}$$

$$\begin{aligned}
&\text{for } i = 1, \dots, a, j = b+1, \dots, q & F_{i,j} &= 0, \\
&\text{for } i = a+1, \dots, m, j = 1, \dots, b & F_{i,j} &= 0, \\
&\text{for } i = a+1, \dots, m, j = b+1, \dots, q & F_{i,j} &= 0.
\end{aligned}$$

Finally, from (18) and (19) i.e. $E = F$ we get that

$$\text{for } i = 1, \dots, a, j = 1, \dots, b \quad c_{i,j} \text{ are arbitrary elements of } \mathbb{C},$$

$$\text{for } i = a+1, \dots, m, j = 1, \dots, b \quad c_{i,j} = \sum_{l=1}^a \alpha_{i,l} c_{l,j},$$

$$\text{for } i = 1, \dots, a, j = b+1, \dots, q \quad c_{i,j} = \sum_{k=1}^b \beta_{k,j} c_{i,k},$$

$$\text{for } i = a+1, \dots, m, j = b+1, \dots, q \quad c_{i,j} = \sum_{l=1}^a \sum_{k=1}^b \alpha_{i,l} \beta_{k,j} c_{l,k}.$$

(\Leftarrow) : Suppose that

$$\widehat{C} = \begin{bmatrix} c_{1,1} & \dots & c_{1,b} & \sum_{k=1}^b \beta_{k,b+1} c_{1,k} & \dots & \sum_{k=1}^b \beta_{k,q} c_{1,k} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ c_{a,1} & \dots & c_{a,b} & \sum_{k=1}^b \beta_{k,b+1} c_{a,k} & \dots & \sum_{k=1}^b \beta_{k,q} c_{a,k} \\ \sum_{l=1}^a \alpha_{a+1,l} c_{l,1} & \dots & \sum_{l=1}^a \alpha_{a+1,l} c_{l,b} & \sum_{l=1}^a \sum_{k=1}^b \alpha_{a+1,l} \beta_{k,b+1} c_{l,k} & \dots & \sum_{l=1}^a \sum_{k=1}^b \alpha_{a+1,l} \beta_{k,q} c_{l,k} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ \sum_{l=1}^a \alpha_{m,l} c_{l,1} & \dots & \sum_{l=1}^a \alpha_{m,l} c_{l,b} & \sum_{l=1}^a \sum_{k=1}^b \alpha_{m,l} \beta_{k,b+1} c_{l,k} & \dots & \sum_{l=1}^a \sum_{k=1}^b \alpha_{m,l} \beta_{k,q} c_{l,k} \end{bmatrix}.$$

Then,

$$\widehat{Q}_1 \widehat{C} \widehat{P}_2 = \dots = \begin{bmatrix} c_{1,1} & \dots & c_{1,b} & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ c_{a,1} & \dots & c_{a,b} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} \quad (20)$$

and

$$\left[\begin{array}{c|c} I_a & X_1 \\ \hline 0 & 0 \end{array} \right] \widehat{Q}_1 \widehat{C} \widehat{P}_2 \left[\begin{array}{c|c} I_b & 0 \\ \hline Y_2 & 0 \end{array} \right] = \dots = \begin{bmatrix} c_{1,1} & \dots & c_{1,b} & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ c_{a,1} & \dots & c_{a,b} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}. \quad (21)$$

From (20) and (21) we conclude that

$$\left[\begin{array}{c|c} I_a & X_1 \\ \hline 0 & 0 \end{array} \right] \widehat{Q_1} \widehat{C} \widehat{P_2} \left[\begin{array}{c|c} I_b & 0 \\ \hline Y_2 & 0 \end{array} \right] = \widehat{Q_1} \widehat{C} \widehat{P_2}.$$

By multiplying the previous equality by $\widehat{Q_1}^{-1}$ on the left and by $\widehat{P_2}^{-1}$ on the right we obtain the following equality:

$$\widehat{Q_1}^{-1} \left[\begin{array}{c|c} I_a & X_1 \\ \hline 0 & 0 \end{array} \right] \widehat{Q_1} \widehat{C} \widehat{P_2} \left[\begin{array}{c|c} I_b & 0 \\ \hline Y_2 & 0 \end{array} \right] \widehat{P_2}^{-1} = \widehat{C}. \quad (22)$$

From (22), considering the equalities (11) and (12), we see that the condition (17) is true. Finally, based on Lemma 2.1. we conclude that the condition (16) is true. ♦

Example 2.1. *Let be given the following matrices:*

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 2 \\ 2 & 1 & -1 \\ -1 & -4 & 3 \\ 3 & -2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 2 & 3 & -1 \\ 0 & 3 & 1 & 4 & 2 \\ 0 & 4 & 1 & 5 & 3 \\ 0 & 2 & 3 & 5 & -1 \end{bmatrix}.$$

Then, $\text{rank}(A)=2$, $\text{rank}(B)=2$ and for

$$T_A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } T_B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

we get that

$$\widehat{A} = T_A A = \dots = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -1 \\ -1 & -4 & 3 \\ 3 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \widehat{B} = B T_B = \dots = \begin{bmatrix} 1 & 2 & 0 & 3 & -1 \\ 3 & 1 & 0 & 4 & 2 \\ 4 & 1 & 0 & 5 & 3 \\ 2 & 3 & 0 & 5 & -1 \end{bmatrix}.$$

Therefore,

$$\widehat{A}_{3 \rightarrow} = \widehat{A}_{1 \rightarrow} - \widehat{A}_{2 \rightarrow}, \quad \widehat{A}_{4 \rightarrow} = \widehat{A}_{1 \rightarrow} + \widehat{A}_{2 \rightarrow}, \quad \widehat{A}_{5 \rightarrow} = 0\widehat{A}_{1 \rightarrow} + 0\widehat{A}_{2 \rightarrow}$$

and

$$\widehat{B}_{\downarrow 3} = 0\widehat{B}_{\downarrow 1} + 0\widehat{B}_{\downarrow 2}, \quad \widehat{B}_{\downarrow 4} = \widehat{B}_{\downarrow 1} + \widehat{B}_{\downarrow 2}, \quad \widehat{B}_{\downarrow 5} = \widehat{B}_{\downarrow 1} - \widehat{B}_{\downarrow 2}.$$

From this we get that

$$\alpha_{3,1} = 1, \alpha_{3,2} = -1, \alpha_{4,1} = 1, \alpha_{4,2} = 1, \alpha_{5,1} = 0, \alpha_{5,2} = 0$$

and

$$\beta_{1,3} = 0, \beta_{2,3} = 0, \beta_{1,4} = 1, \beta_{2,4} = 1, \beta_{1,5} = 1, \beta_{2,5} = -1.$$

Based on Theorem 2.1. each matrix \hat{C} which has the following form

$$\hat{C} = \begin{bmatrix} c_{1,1} & c_{1,2} & 0 & c_{1,1} + c_{1,2} & c_{1,1} - c_{1,2} \\ c_{2,1} & c_{2,2} & 0 & c_{2,1} + c_{2,2} & c_{2,1} - c_{2,2} \\ c_{1,1} - c_{2,1} & c_{1,2} - c_{2,2} & 0 & c_{1,1} - c_{2,1} + c_{1,2} - c_{2,2} & c_{1,1} - c_{2,1} - c_{1,2} + c_{2,2} \\ c_{1,1} + c_{2,1} & c_{1,2} + c_{2,2} & 0 & c_{1,1} + c_{2,1} + c_{1,2} + c_{2,2} & c_{1,1} + c_{2,1} - c_{1,2} - c_{2,2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

satisfies the condition (17). From that we conclude that each matrix $C = T_A^{-1} \hat{C} T_B^{-1}$ which has the following form

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & c_{1,1} & c_{1,2} & c_{1,1} + c_{1,2} & c_{1,1} - c_{1,2} \\ 0 & c_{2,1} & c_{2,2} & c_{2,1} + c_{2,2} & c_{2,1} - c_{2,2} \\ 0 & c_{1,1} - c_{2,1} & c_{1,2} - c_{2,2} & c_{1,1} - c_{2,1} + c_{1,2} - c_{2,2} & c_{1,1} - c_{2,1} - c_{1,2} + c_{2,2} \\ 0 & c_{1,1} + c_{2,1} & c_{1,2} + c_{2,2} & c_{1,1} + c_{2,1} + c_{1,2} + c_{2,2} & c_{1,1} + c_{2,1} - c_{1,2} - c_{2,2} \end{bmatrix}$$

satisfies the condition (16).♦

In the following example we will show one useful application of Theorem 2.1. Namely, the result in Theorem 2.1. can be applied to determine the consistency of the matrix equation (2).

Example 2.2. Let A and B are the matrices as in Example 2.1. and

$$a) C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & -2 & 2 & 0 & -4 \\ 0 & 3 & -2 & 1 & 5 \\ 0 & -1 & 2 & 1 & -3 \end{bmatrix}, \quad b) C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & -2 & 2 & 0 & 4 \\ 0 & 3 & -2 & 1 & 5 \\ 0 & -1 & 2 & 1 & -3 \end{bmatrix}.$$

If we compare the matrix C from a) and from b) with the general form of the matrix C which satisfies the condition (16) (see Example 2.1.) we see that the matrix C from a) satisfies the condition (16) and the matrix C from b) does not satisfy the condition (16). Therefore, the matrix equation (2) is consistent for the matrix C from a) and it is not consistent for the matrix C from b).♦

2.2. Recall that the matrix equation $AXB = C$ is marked with (2). In the paper [1] R. Penrose proved the following theorem related to the matrix equation (2).

Theorem 2.2. The matrix equation (2) is consistent iff for some choice of $\{1\}$ -inverses $A^{(1)}$ and $B^{(1)}$ of the matrices A and B the condition (16) is true. The general solution of the matrix equation (2) is given by the formula

$$X = f(Y) = A^{(1)}CB^{(1)} + Y - A^{(1)}AYBB^{(1)}, \quad (23)$$

where Y is an arbitrary matrix corresponding dimensions.♦

Remark 2.1. *If the matrix equation (2) is consistent, the equivalence*

$$AXB = C \iff X = f(X) = X - A^{(1)}(AXB - C)B^{(1)} \quad (24)$$

is true. Therefore, the starting equation is equivalent to some reproductive equation. Based on Theorem 1.2. we can also conclude that (23) is the general solution of the matrix equation (2).◇

In this paper we give an extension of Theorem 2.2.

Theorem 2.3. *If X_0 is any particular solution of the matrix equation (2), the general solution of the matrix equation (2) is given by the formula*

$$X = g(Y) = X_0 + Y - A^{(1)}AYBB^{(1)}, \quad (25)$$

where Y is an arbitrary matrix corresponding dimensions. The function g satisfies the condition of reproductivity (1) iff $X_0 = A^{(1)}CB^{(1)}$.

Proof. It is easily to see that the solution of the matrix equation (2) is given by (25). On the contrary, let X is any solution of the matrix equation (2), then

$$\begin{aligned} X &= X - A^{(1)}CB^{(1)} + A^{(1)}CB^{(1)} \\ &= X - A^{(1)}AXBB^{(1)} + A^{(1)}AX_0BB^{(1)} \\ &= X - A^{(1)}A(X - X_0)BB^{(1)} \\ &= X_0 + (X - X_0) - A^{(1)}A(X - X_0)BB^{(1)} \\ &= X_0 + Y - A^{(1)}AYBB^{(1)} = g(Y), \end{aligned}$$

where $Y = X - X_0$. From this we see that every solution X of the matrix equation (2) can be represented in the form (25).

Based on the following matrix equality

$$g^2(Y) = g(Y) + (X_0 - A^{(1)}CB^{(1)})$$

we see that the function g satisfies the condition (1) iff $X_0 = A^{(1)}CB^{(1)}$.◇

So, the general solution (25) of the matrix equation (2) is reproductive iff $X_0 = A^{(1)}CB^{(1)}$. Therefore, Penrose's general solution (23) of the matrix equation (2) is the reproductive solution.

Remark 2.2. *If the condition (16) is not true, the matrix equation (2) is solved approximately as described in [26], [31] and [35].◇*

2.3. Using the form of the matrix \hat{C} which is necessary and sufficient for satisfying the condition (17) we give the necessary and sufficient form for a particular solution X_0 of the matrix equation (2) so that the general solution (25) of the matrix equation (2) is reproductive.

Theorem 2.4. Let X_0 is any particular solution of the matrix equation (2). The general solution (25) of the matrix equation (2) is reproductive iff

$$X_0 = P_1 \left[\begin{array}{c|c} C_1 & C_1 Y_1 \\ \hline X_2 C_1 & X_2 C_1 Y_1 \end{array} \right] Q_2$$

where P_1, Q_2, X_2, Y_1 are the matrices from (10) and C_1 is the submatrix of the matrix \widehat{C} and it has the following form:

$$C_1 = \begin{bmatrix} c_{1,1} & \dots & c_{1,b} \\ \cdot & \dots & \cdot \\ \cdot & \dots & \cdot \\ \cdot & \dots & \cdot \\ c_{a,1} & \dots & c_{a,b} \end{bmatrix},$$

where $c_{i,j}$ are some elements of \mathbb{C} .

Proof. The statement follows from Theorem 2.3. because

$$\begin{aligned} X_0 &= A^{(1)}CB^{(1)} \underset{(10)}{=} P_1 \left[\begin{array}{c|c} I_a & X_1 \\ \hline X_2 & X_3 \end{array} \right] Q_1 C P_2 \left[\begin{array}{c|c} I_b & Y_1 \\ \hline Y_2 & Y_3 \end{array} \right] Q_2 \\ &\stackrel{(5)}{=} P_1 \left[\begin{array}{c|c} I_a & X_1 \\ \hline X_2 & X_3 \end{array} \right] Q_1 T_A^{-1} \widehat{C} T_B^{-1} P_2 \left[\begin{array}{c|c} I_b & Y_1 \\ \hline Y_2 & Y_3 \end{array} \right] Q_2 \\ &\stackrel{(8)}{=} P_1 \left[\begin{array}{c|c} I_a & X_1 \\ \hline X_2 & X_3 \end{array} \right] \widehat{Q}_1 \widehat{C} \widehat{P}_2 \left[\begin{array}{c|c} I_b & Y_1 \\ \hline Y_2 & Y_3 \end{array} \right] Q_2 \\ &\stackrel{(20)}{=} P_1 \left[\begin{array}{c|c} I_a & X_1 \\ \hline X_2 & X_3 \end{array} \right] \left[\begin{array}{c|c} C_1 & 0 \\ \hline 0 & 0 \end{array} \right] \left[\begin{array}{c|c} I_b & Y_1 \\ \hline Y_2 & Y_3 \end{array} \right] Q_2 \\ &= P_1 \left[\begin{array}{c|c} C_1 & C_1 Y_1 \\ \hline X_2 C_1 & X_2 C_1 Y_1 \end{array} \right] Q_2. \blacklozenge \end{aligned}$$

Remark 2.3. In the paper [36] authors proved that there is a particular solution X_0 of the matrix equation (2) so that $X_0 \neq A^{(1)}CB^{(1)}$ for any choice of $\{1\}$ -inverses $A^{(1)}$ and $B^{(1)}$. \diamond

2.4. In this part of the paper we analysed solutions of some matrix systems using the concept of reproductivity.

Application 2.1. In [1] R. Penrose studied a matrix system

$$(26a) \quad AX = B \quad \wedge \quad (26b) \quad XD = E, \quad (26)$$

where A, B, D and E are given complex matrices corresponding dimensions. He proved that

$$X_1 = A^{(1)}B + ED^{(1)} - A^{(1)}AED^{(1)} \quad (27)$$

is one common solution of the matrix equations (26a) and (26b) provided $AE = BD$ and the matrix equations (26a) and (26b) are consistent.

In [21] A. Ben-Israel and T.N.E. Greville proved that the matrix equations (26a) and (26b) have a common solution iff each equation separately has a solution and $AE = BD$. Also, they proved that if X_0 is any common solution of the matrix equations (26a) and (26b), the general solution of the matrix system (26) is given by the formula

$$X = g(Y) = X_0 + (I - A^{(1)}A)Y(I - DD^{(1)}), \quad (28)$$

where Y is an arbitrary matrix corresponding dimensions.

By multiplying the matrix equation (26a) by D on the right and the matrix equation (26b) by A on the left we get

$$AXD = BD, \quad AXD = AE. \quad (29)$$

From this we conclude that $AE = BD$.

We will prove that if the matrix system (26) is consistent, the general reproductive solution is given by the formula

$$X = f(Y) = A^{(1)}B + ED^{(1)} - A^{(1)}AED^{(1)} + (I - A^{(1)}A)Y(I - DD^{(1)}), \quad (30)$$

where Y is an arbitrary matrix corresponding dimensions.

If the matrix system (26) is consistent, the following equivalence is true.

$$(AX = B \wedge XD = E) \iff X = f(X) \quad (31)$$

The direct implication of (31) follows by the following implications (see Remark 2.1.):

$$\begin{aligned} AX = B &\implies X = f_1(X) = A^{(1)}B + X - A^{(1)}AX, \\ XD = E &\implies X = f_2(X) = ED^{(1)} + X - XDD^{(1)}, \\ AXD = BD = AE &\implies X = f_3(X) = A^{(1)}AED^{(1)} + X - A^{(1)}AXDD^{(1)}. \end{aligned}$$

From the previous implications we get that

$$(AX = B \wedge XD = E) \implies X = f(X) = f_1(X) + f_2(X) - f_3(X).$$

The reverse implication of (31) is trivial. Notice that the function f is reproductive. Therefore, if the matrix system (26) is consistent, it is equivalent to the reproductive matrix equation $X = f(X)$. Based on Theorem 1.2. we conclude that $X = f(Y)$ is the general reproductive solution of the matrix system (26). It is easy to see that if there is a particular solution X_0 so that $X_0 \neq X_1$, then $X = g(Y)$ is the general non-reproductive solution. \diamond

Application 2.2. Let $A \in \mathbf{C}^{n \times n}$ be a singular matrix. In this section we consider a matrix system

$$AXA = A \quad \wedge \quad AX = XA. \quad (32)$$

The consistency of the matrix system (32) is determined by Theorem 1 in [11]. Let \bar{A} is commutative $\{1\}$ -inverse. Based on the reproductivity, we give a new proof that the formula

$$X = f(Y) = Y + \bar{A}A\bar{A} - \bar{A}AY - YAA + \bar{A}AY\bar{A}, \quad (33)$$

where Y is an arbitrary matrix corresponding dimensions, represents the general solution of the matrix system (32). The formula (33) is listed by Theorem 3 in the paper [11].

If the matrix system (32) is consistent, the equivalence

$$(AXA = A \wedge AX = XA) \iff X = f(X) \quad (34)$$

is true.

The direct implication of (34) is based on the following equalities:

$$\underbrace{\bar{A}A}_{(=AXA)} \underbrace{\bar{A}}_{(=\bar{A}A)} = \bar{A}AX \underbrace{A\bar{A}}_{(=XA)} = \bar{A}AX \bar{A}A = \bar{A}X \underbrace{A\bar{A}A}_{(=A)} = \bar{A} \underbrace{XA}_{(=AX)} = \bar{A}AX$$

and

$$\underbrace{\bar{A}A}_{(=A\bar{A})} X A \bar{A} = A \bar{A} \underbrace{XA}_{(=AX)} \bar{A} = \underbrace{A\bar{A}AX}_{(=A)} \bar{A} = \underbrace{AX}_{(=XA)} \bar{A} = X A \bar{A}.$$

From this we get that $X = X + \bar{A}A\bar{A} - \bar{A}AX + \bar{A}AXA\bar{A} - XA\bar{A} = f(X)$. The reverse implication of (34) is trivial. Notice that the function f is reproductive. Therefore, if the matrix system (32) is consistent, it is equivalent to the reproductive matrix equation $X = f(X)$. Based on Theorem 1.2. we conclude that $X = f(Y)$ is the general reproductive solution of the matrix system (32). If X_0 is any solution of the matrix system (32), the formula

$$X = g(Y) = Y + X_0 - \bar{A}AY - YAA + \bar{A}AY\bar{A}, \quad (35)$$

also determines the general solution of the matrix system (32) because the equality $g(Y) = f(Y + X_0)$ is true. If there is a particular solution X_0 of the matrix system (32) so that $X_0 \neq \bar{A}A\bar{A}$, then $X = g(Y)$ is the general non-reproductive solution. The previously described process of proving the generality of the solutions can also be applied to the matrix system which determines k -commutative $\{1\}$ -inverse from [10], [11], [15]. \diamond

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